THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. V. NOVEMBER, 1898. No. 11.

POINT INVARIANTS FOR THE FINITE CONTINUOUS GROUPS OF THE PLANE.

By DAVID A. ROTHROCK, Indiana University, Bloomington, Indiana.

In an article entitled "Theorie der Transformationsgruppen," Mathematische Annalen, Bd. XVI., Lie has classified and reduced to canonical forms all the *finite continuous* groups of the plane. We may very appropriately call these the *Lie groups*. In the present paper it is proposed to determine the functions of the coördinates of n points which remain invariant by these groups. In some cases the computations are quite complex; only the methods of calculation are given.

An infinitesimal point transformation gives to x and y the increments

$$\delta x = \xi(x, y)\delta t, \quad \delta y = \eta(x, y)\delta t,$$

respectively, where δt is independent of x and y. Such an infinitesimal transformation is denoted in the Lie notation by

$$Xf \equiv \xi(x, y), \frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}. \tag{1}$$

The variation of any function $\varphi(x, y)$ by this transformation is given by $X\varphi(x, y)\delta t$. Hence, if $\varphi(x, y)$ is to remain invariant by the transformation Xf, its variation must be zero, or

$$X\varphi \equiv \xi(x,\ y)\frac{\partial \varphi}{\partial x} + \eta(x,\ y)\frac{\partial \varphi}{\partial y} = 0.$$

The function $\varphi(x, y)$, invariant by the infinitesimal transformation Xf, is determined as a solution of the linear partial differential equation $Xf=0$.

The infinitesimal transformation Xf may be *extended* to include the increments of any number of points, x_i, y_i, $i=1, 2 \ldots\ldots n$. We may write this extended transformation as

$$Wf \equiv \sum_1^n iX^{(i)}f \equiv \sum_1^n i\xi(x_i,\ y_i)\ \frac{\partial f}{\partial x_i} + \sum_1^n i\eta(x_i,\ y_i)\ \frac{\partial f}{\partial y_i}. \tag{2}$$

The functions of the coördinates of n points invariant by Wf will be the $2n-1$ independent solutions of $Wf=0$. n of these solutions may be selected in the form $\varphi(x_i,\ y_i)$, where $\varphi(x, y)$ is a solution of $Xf=0$; the remaining $n-1$ solutions will in general differ from $\varphi(x, y)$ in form.*

Infinitesimal transformations are called *independent* when no one can be expressed as a linear function of the others with constant coefficients. r such independent infinitesimal transformations

$$X_k f \equiv \xi_k(x,\ y)\frac{\partial f}{\partial x} + \eta_k(x, y)\frac{\partial f}{\partial y},\ (k=1\ \ldots\ldots r),$$

form a group when

$$X_i(X_k f) - X_k(X_i f) \equiv \sum_1^r s c_{iks} X_s f,\ (c_{iks} = \text{constants}). \tag{3}$$

The transformations of this *r-parameter* group $X_k f$ may be extended after the manner of (2) above, giving us $W_k f$ which determine the increments of a function $f(x_1, y_1;\ x_2, y_2;\ \ldots\ldots x_n,\ y_n)$. On account of relation (3),

$$W_1 f=0,\ \ W_2 f=0,\ \ldots\ldots\ W_r f=0$$

are known to form a *complete* system of linear partial differential equations in $2n$ variables $x_1 y_1$, $x_2 y_2$, $\ldots\ldots x_n y_n$, with at least $2n-r$ independent solutions. If the r-rowed determinants of the coefficients vanish, more than $2n-r$ independent solutions will exist. These solutions are the invariants of the coördinates of n points by the r-parameter group of $X_k f$.

According to the theory here outlined we shall determine the *point-invariants* of the finite continuous groups as classified by Lie in the Mathematische Annalen, Vol. XVI.

*Lie: Theorie der Transformationsgruppen, Bd. I., §59.

§ 1.
Invariants of the Primitive Groups.

The *primitive* groups of the plane leave *no* family of ∞^1 curves invariant, and may be reduced by a proper choice of variables to some one of the canonical forms known as (1) special linear, (2) general linear, (3) general projective.

1. *The special linear group*

$$\boxed{p,\ q,\ xq,\ xp-yq,\ yp}.$$

Here $p=\frac{\partial f}{\partial x}$, $q=\frac{\partial f}{\partial y}$. The increments of $f(x_i, y_i)$ by the members of this group *extended* are given by

$$W_1 f\equiv\sum_1^n i\frac{\partial f}{\partial x_i},\quad W_2 f\equiv\sum_1^n i\frac{\partial f}{\partial y_i},\quad W_3 f\equiv\sum_1^n ix_i\frac{\partial f}{\partial y_i},$$

$$W_4 f\equiv\sum_1^n i\left(x_i\frac{\partial f}{\partial x_i}-y_i\frac{\partial f}{\partial y_i}\right),\quad W_5 f\equiv\sum_1^n iy_i\frac{\partial f}{\partial x_i}.$$

The invariant functions sought will be the $2n-5$ independent solutions of the complete system

$$W_1 f=0,\quad W_2 f=0,\quad W_3 f=0,\quad W_4 f=0,\quad W_5 f=0. \tag{s}$$

The first two equations show the solutions to be functions of

$$\varphi_j=x_1-x_j,\quad \psi_j=y_1-y_j,\quad (j=2\ldots\ldots n).$$

The remaining equations then take the form

$$\sum_2^n j\varphi_j\frac{\partial f}{\partial \psi_j}=0,\quad \sum_2^n j\left(\varphi_j\frac{\partial f}{\partial \varphi_j}-\psi_j\frac{\partial f}{\partial \psi_j}\right)=0,\quad \sum_2^n j\psi_j\frac{\partial f}{\partial \varphi_j}=0.$$

The second of these equations has solutions

$$\Phi_j=\varphi_j\psi_j,\quad \Psi_k=\varphi_2\psi_k,\quad (k=3\ldots\ldots n).$$

With these functions as new variables, the first and third equations become

$$\frac{\partial f}{\partial \phi_2}+\sum_3^n k\frac{\phi_k}{\Psi_k}\left\{\frac{\phi_k}{\Psi_h}\frac{\partial f}{\partial \phi_k}+\frac{\partial f}{\partial \psi_k}\right\}=0, \tag{1}$$

$$\frac{\phi_2}{\Psi_3}\frac{\partial f}{\partial \phi_2}+\frac{1}{\phi_2\psi_3}\sum_3^n k\psi_k^2\frac{\partial f}{\partial \varphi_k}+\frac{1}{\psi_3}\sum_3^n k\psi_k\frac{\partial f}{\partial \psi_k}=0. \tag{2}$$

The solutions of (1) are seen to be

$$\sigma_k = \frac{\psi_k}{\varphi_k}, \ \rho_k = \varphi_2 - \frac{\psi_k^2}{\varphi_k},$$

while (2) takes the form

$$\sum_3^n k\left(\sigma_k\rho_k\frac{\partial f}{\partial \sigma_k} + \rho_k^2\frac{\partial f}{\partial \rho_k}\right) = 0,$$

with solutions

$$\Delta_k = -\frac{\rho_k}{\sigma_k}, \ I_e = \frac{1}{\rho_e} - \frac{1}{\rho_s}, \ (k=3 \ldots\ldots n, \ l=4 \ldots\ldots n).$$

Since any functions of Δ and I will be solutions of our complete system (s), we may choose

$$\Delta_k = -\frac{\rho_k}{\rho_2} = |\ 1\ 2\ k\ |, \ D_l = I_l.\Delta_3.\Delta_l = |\ 1\ 3\ l\ |, \ \text{where } |\ i\ j\ k\ | = \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix},$$

as solutions of (s), and, therefore, as the $2n-5$ invariants of the coördinates of n points.

The forms of Δ and D show that *the special linear group leaves invariant all areas.*

2. *The general linear group*

$$\boxed{p, \ q, \ xq, \ xp-yq, \ yp, \ xp+yq}$$

This group furnishes a complete system of six linear partial differential equations, the first five equations of the system being identical with those of the preceding section. We need only to determine the functions of Δ_k and D_l which satisfy

$$\sum_1^n i\left(x_i\frac{\partial f}{\partial x_i} + y_i\frac{\partial f}{\partial y_i}\right) = 0.$$

This equation requires that x, y enter in the final solutions with the degree zero. Hence, we may write at once the invariants :

$$I_l = \frac{\Delta_l}{\Delta_3} = |\ 1\ 2\ l\ | : |\ 1\ 2\ 3\ |$$

$$J_l = \frac{D_l}{\Delta_3} = |\ 1\ 3\ l\ | : |\ 1\ 2\ 3\ | \quad (l=4, \ldots\ldots n).$$

Hence, *by the general linear group the ratio of areas remain constant.*

3. *The general projective group*

$$\boxed{p, \ q, \ xq, \ xp-yq, \ yp, \ xp+yq, \ x^2p+xyq, \ xyp+y^2q}.$$

The members of this group extended and equated to zero give a complete system of eight partial differential equations, the first six of which are identical with those of the general linear group, and therefore have solutions I, J defined above. The last two equations,

$$\sum_1^n ix_i(x_i\frac{\partial f}{\partial x_i} + y_i\frac{\partial f}{\partial y_i})=\sum_1^n iy_i(x_i\frac{\partial f}{\partial x_i} + y_i\frac{\partial f}{\partial y_i})=0,$$

expressed in terms of I and J, become somewhat complex. For brevity we shall write these symbolically as

$$Uf=0, \quad Vf=0,$$

where $Uf=0$ has the solutions

$$I_4, \quad \Phi_m=\frac{I_m J_4}{J_m}, \quad \Psi_m=\frac{\phi_m+I_m(I_4-\phi_m-1)}{I_m(J_4-I_4+1)}, \quad (m=5.\dots.n).$$

With I_4, ϕ_m and ψ_m as new variables, the equation $Vf=0$ takes the simple form

$$I_4\frac{\partial f}{\partial I_4}+\sum_5^n m\varphi_m\frac{\partial f}{\partial \varphi_m}=0,$$

whose solutions are clearly

$$\theta_m=\frac{\varphi_m}{I_4}, \quad \Psi_m, \quad (m=5\dots\dots n).$$

Selecting as invariants θ_m and $H_m=\dfrac{1+\Psi_m}{\theta_m}$, and restoring the variables x_i, y_i we have

$$\theta_m=\frac{|\,1\;2\;m\,|\,.\,|\,1\;3\;4\,|}{|\,1\;2\;4\,|\,.\,|\,1\;3\;m\,|}, \quad H_m=\frac{|\,1\;2\;4\,|\,.\,|\,2\;3\;m\,|}{|\,1\;2\;m\,|\,.\,|\,2\;3\;4\,|}.$$

The forms of θ and H show *that the general projective group leaves invariant the cross-ratios of five points*. Five points have two independent invariants, θ, H; four points have no invariant unless they be collinear, in which case the invariant is the cross-ratio of the four points.

§ 2.

Invariants of such Imprimitive Groups as Leave Unchanged One Family of ∞^1 Curves.

The remaining finite continuous groups of the plane are known as *imprimitive*, and are classified according as they leave invariant *one*, *two*, or an *infinite number* of families of ∞^1 curves.* The groups of the first category have been

*Lie: Math. Annalen, Bd. XVI.

reduced by Lie to such canonical forms that the family of curves x=constant is transformed into itself.

4. $$\boxed{\begin{array}{c} X_1 q,\ X_2 q,\ X_3 q,\ \ldots\ldots X_r q \\ r>1 \end{array}}\ .$$

Here X_k is a function of x alone. This group leaves the curves of the family $x=a$ singly invariant.

The complete system

$$W_k f \equiv \sum_1^n i X_k(x_i) \frac{\partial f}{\partial y_i} = 0, \quad (k=1 \ \ldots\ r),$$

corresponding to this group, has as solutions x_1, x_2, ... x_n and $n-r$ other independent functions Δ_s, ($s=1 \ldots\ldots n-r$), which we shall define as the $n-r$ determinants of the matrix

$$\left\| \begin{array}{ccccc} y_1 & y_2 & y_3 & \cdots y_{r+s} & \cdots y_n \\ X_1(x_1) & \cdots & \cdots & \cdots & \cdots \\ X_2(x_1) & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ X_r(x_1) & \cdots & \cdots & \cdots & \cdots \end{array} \right\|, \qquad (M_r)$$

$$(n \gtreqless r+1)$$

formed by filling the $(r+1)$th column successively by the $(r+1)$, the $(r+2)$...., and the nth.

Our invariants are x_i and Δ_s, ($i=1 \ldots\ldots n$, $s=1 \ldots\ldots n-r$).

5. $$\boxed{\begin{array}{c} X_1 q,\ X_2 q \ldots\ldots X_{r-1} q,\ yq \\ r>2 \end{array}}\ .$$

This group furnishes the complete system

$$W_k f \equiv \sum_1^n i X_k(x_i) \frac{\partial f}{\partial y_i} = 0, \quad Yf \equiv \sum_1^n i y_i \frac{\partial f}{\partial y_i} = 0, \quad (k=1 \ldots\ldots r-1).$$

The solutions of $W_k f=0$ are clearly x_1, x_2, ...x_n and the determinants Δ_s, ($s=0$, 1, ...$n-r$), of the matrix (M_{r-1}) defined above. $Yf=0$ requires the ratios of y_i to appear. We may then write as invariants of the group

$$x_1,\ x_2,\ \ldots\ldots x_n \text{ and } \rho_t = \frac{\Delta_t}{\Delta_0}, \quad (t=1 \ldots\ldots n-r).$$

6. $$\boxed{\begin{array}{c} \varepsilon^{\alpha_k x}q,\ x\varepsilon^{\alpha_k x}q,\ x^2\varepsilon^{\alpha_k x}q,\ \ldots\ldots x^{\rho_k}\varepsilon^{\alpha_k x}q,\ p \\ k=1\ldots\ldots m,\ \sum_1^m k\rho_k+m=r-1,\ r>2 \end{array}}$$

We have

$$W_k^{t_k}f\equiv\sum_1^n i(x_i)^{t_k}\,.\,\varepsilon^{\alpha_k x_i}\frac{\partial f}{\partial y_i}=0,\ (k=1,\ \ldots\ldots m,\ t_k=0,\ 1\ldots\ldots\rho_k).$$

$$Xf\equiv\sum_1^n i\frac{\partial f}{\partial x_i}=0.$$

The last equation requires the functions

$$\varphi_j=x_j-x_1\quad (j=2\ldots\ldots n),$$

to appear in the solutions of the system. On dividing the remaining equations, respectively, by $\varepsilon^{\alpha_t x_1}$, the exponents of ε all become functions of φ_j. The independent determinants Δ_s, $(s=0,\ 1\ldots\ldots n-r)$ of the matrix (M_{r-1}), formed as in 4, will be solutions.

The invariants are, therefore, φ_j and Δ_s.

7. $$\boxed{\begin{array}{c} \varepsilon^{\alpha_k x}q,\ x\varepsilon^{\alpha_k x}q,\ x^2\varepsilon^{\alpha_k x}q,\ \ldots\ldots\ x^{\rho_k}\varepsilon^{\alpha_k x}q,\ yq,\ p \\ k=1,\ \ldots\ldots\ m,\ \sum_1^n k\rho_k+m=r-2,\ r>3 \end{array}}.$$

The complete system given by this group is

$$W_k{}^{t_k}f\equiv\sum_1^n i(x_i)^{t_k}.\varepsilon^{\alpha_k x_i}\frac{\partial f}{\partial y_i}=0,\ (k=1\ldots\ldots m,\ t_k=0,\ 1,\ \ldots\ldots\rho_k).$$

$$Yf\equiv\sum_1^n iy_i\frac{\partial f}{\partial y_i}=0,\ Xf\equiv\sum_1^n i\frac{\partial f}{\partial x_i}=0.$$

If a matrix be constructed, as indicated in 4 and 6, from the coefficients of the first $r-2$ equations, it will be observed that the independent determinants Δ_s, $(s=-1,\ 0,\ 1\ldots\ldots n-r)$, will be linear and homogeneous in y_i with coefficients composed of functions of $\varphi_j=x_j-x_1$. Δ_s will be solutions of all equations except $Yf=0$, which requires the ratios of y_i to appear. Hence, the invariants may be written

$$\varphi_j=x_j-x_1,\ \rho_t=\frac{\Delta_t}{\Delta_{-1}},\ (j=2\ldots\ldots n,\ t=0\ldots\ldots\ n-r).$$

8. $$\boxed{\begin{matrix} q, \quad xq, \quad x^2q, \ldots\ldots x^{r-3}q, \quad p, \quad xp+cyq \\ r>3 \end{matrix}}.$$

Here the complete system is

$$W_k f \equiv \sum_1^n i x_i^k \frac{\partial f}{\partial y_i}=0, \quad (k=0,\ 1 \ \ldots\ldots r-3)$$

$$Yf \equiv \sum_1^n i x_i \frac{\partial f}{\partial x_i}+c\sum_1^n i y_i \frac{\partial f}{\partial y_i}=0, \quad Xf \equiv \sum_1^n i \frac{\partial f}{\partial x_i}=0.$$

The solutions

$$\psi_j=y_1-y_j, \quad \varphi_j=x_1-x_j, \quad (j=2\ldots\ldots n),$$

of the first and last of these equations, introduced in Yf, give

$$Y'f \equiv \sum_2^n j\,(\varphi_j \frac{\partial f}{\partial \varphi_j}+c\,\psi_j \frac{\partial f}{\partial \psi_j})=0,$$

whose solutions are

$$\sigma_k=\frac{\varphi_k}{\varphi_2}, \quad \rho_j=\frac{\psi_j}{(\varphi_j)^c}, \quad (k=3\ldots\ldots n,\ j=2\ldots\ldots n).$$

$W_k f$, expressed in σ, ρ, is

$$W_t'f \equiv \frac{\partial f}{\partial \rho_2}+\sum_3^n k\sigma_k^{t-c}\frac{\partial f}{\partial \rho_k}=0, \quad (t=1,\ \ldots\ldots r-3).$$

The solutions of these $r-3$ equations are σ_k, and the determinants Δ_s, $(s=1,\ 2\ldots\ldots n-r+2)$, of the matrix

$$\left\| \begin{matrix} \rho_2 & \rho_3 & \rho_4 \ldots\ldots \rho_{r-3+s} \ldots\ldots & \rho_n \\ 1 & \sigma_3^{1-c} & \sigma_4^{1-c} \ldots\ldots\ldots\ldots & \ldots \\ \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots \\ \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots \\ 1 & \sigma_3^{r-3-c} & \sigma_4^{r-3-c} \ldots\ldots\ldots\ldots & \sigma_n^{r-3-c} \end{matrix} \right\|.$$

Our invariants are

$$\sigma_k=\frac{x_1-x_k}{x_1-x_2} \text{ and } \Delta_s.$$

9. $$\boxed{\begin{matrix} q, \quad xq, \quad x^2q, \ldots\ldots x^{r-3}q, \quad p, \quad xp+[(r-2)y+x^{r-2}]q \\ r>2 \end{matrix}}.$$

The solutions of the complex system,

$$W_k f \equiv \sum_1^n i\, x_i^k \frac{\partial f}{\partial y_i} = 0, \quad (k=0,\ 1 \ldots\ldots r-3),$$

$$Yf \equiv \sum_1^n i \{ x_i \frac{\partial f}{\partial x_i} + [\,(r-2)y_i + x_i^{r-2}] \frac{\partial f}{\partial y_i} \} = 0,$$

$$Xf \equiv \sum_1^n i \frac{\partial f}{d x_i} = 0,$$

may be obtained in a manner similar to 8. The solutions, ψ_j, φ_j, of the first and last equations, introduced in Yf as variables, give

$$Y'f \equiv \sum_2^n j \{ \varphi_j \frac{\partial f}{\partial \varphi_j} + [(r-2)\psi_j + \varphi_j^{r-2}] \frac{\partial f}{\partial \psi_j} \} = 0,$$

with solutions,

$$\sigma_k = \frac{\varphi_k}{\varphi_2}, \quad \rho_j = \log \varphi_j - \frac{\psi_j}{\varphi_j^{r-2}}, \quad (k=3 \ldots\ldots n,\ j=2 \ldots\ldots n).$$

Introducing σ, ρ as new variables in Wf, we have

$$W_t' f \equiv \frac{\partial f}{\partial \rho_2} + \sum_3^n k \sigma_k^{t+2-r} . \frac{\partial f}{\partial \rho_k} = 0,$$

whose solutions are σ_k and Δ_s of the matrix constructed as in 8.

The invariants are, therefore,

$$\sigma_k = \frac{x_1 - x_k}{x_1 - x_2}, \quad \Delta_s, \quad (k=3 \ldots\ldots n, \quad s=1 \ldots\ldots n-r+2).$$

10. $\boxed{\begin{array}{c} q, \quad xq, \quad x^2 q, \quad \ldots\ldots x^{r-4} q, \quad yq, \quad p, \quad xp \\ r>3 \end{array}}$.

For this group

$$W_t f \equiv \sum_1^n i x_i^t \frac{\partial f}{\partial y_i} = 0, \quad (t=0,\ 1 \ldots\ldots r-4),$$

$$Yf \equiv \sum_1^n i y_i \frac{\partial f}{\partial y_i} = 0, \quad X_1 f \equiv \sum_1^n i \frac{\partial f}{\partial x_i} = 0, \quad X_2 f \equiv \sum_1^n i x_i \frac{\partial f}{\partial x_i} = 0.$$

The last two equations show that the ratios of the differences of the x's, say

$$\sigma_k=\frac{x_1-x_k}{x_1-x_2},\ (k=3,\ \ldots\ldots n),$$

shall appear in the final solutions. The $n-r+3$ independent determinants Δ_s, $(s=0,\ 1\ldots\ldots n-r+2)$, of the matrix

$$\left\|\begin{array}{llllll} y_1 & y_2 & y_3\cdots\cdots & y_{r-2+s}\cdots\cdots & & y_n \\ 1 & 1 & 1\ldots\ldots & 1\ldots\ldots\ldots\ldots & & 1 \\ x_1 & x_2 & x_3\ldots\ldots\ldots\ldots\ldots\ldots & & & x_n \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & & \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & & \ldots \\ x_1^{r-4} & x_2^{r-4}\ldots\ldots & \ldots\ldots x_{r-2+s}^{r-4} & \ldots\ldots & & x_n^{r-4} \end{array}\right\|$$

are solutions of the first $r-3$ equations. These determinants are at the same time homogeneous in y_i and x_i-x_k; their ratios will, therefore, satisfy the requirements of

$$\sigma_k=\frac{x_1-x_k}{x_1-x_2}\ \text{and}\ Yf=0.$$

Hence, we may write our $2n-r$ invariants as

$$\sigma_k\ \text{and}\ \rho_t=\frac{\Delta_t}{\Delta_0},\ (k=3\ldots\ldots n,\ t=1\ \ldots\ldots n-r+2).$$

11. $\boxed{q,\quad xq,\quad x^2q,\ \ldots\ldots x^{r-4}q,\quad p,\quad 2xp+(r-4)yq,\quad x^2p+(r-4)xyq \quad r>4}$.

From this group we obtain the differential equations

$$W_tf\equiv\sum_1^n ix_i^t\frac{\partial f}{\partial y_i}=0,\ (t=0,\ 1\ldots\ldots r-4),$$

$$X_1f\equiv\sum_1^n i(2x_i\frac{\partial f}{\partial x_i}+(r-4)\,y_i\frac{\partial f}{\partial y_i})=0,$$

$$X_2f\equiv\sum_1^n i\ (x_i^2\frac{\partial f}{\partial x_i}+(r-4)\ x_iy_i\frac{\partial f}{\partial y_i})=0,$$

$$X_0f\equiv\sum_1^n i\frac{\partial f}{\partial x_i}=0.$$

With the solutions ψ_j, φ_j, of the first and last of these equations, X_2f takes the form

$$X_2{}'f\equiv\sum_2^n j(\varphi_j^2\frac{\partial f}{\partial\varphi_j}+(r-4)\ \varphi_j\psi_j\frac{\partial f}{\partial\psi_j})=0,$$

whose solutions may be selected,

$$u_k=\frac{1}{\varphi_2}-\frac{1}{\varphi_k},\ v_j=\frac{\psi_j}{\varphi_j^{r-4}},\ (k=3\ldots\ldots n,\ j=2\ldots\ldots n).$$

$X_1 f$ then becomes

$$X_1' f\equiv 2\sum_3^n k u_k\frac{\partial f}{\partial u_k}+(r-4)\sum_2^n j v_j\frac{\partial f}{\partial v_j}=0,$$

with solutions

$$\sigma_l=\frac{u_l}{u_3},\quad \rho_k=\frac{v_k}{u_k^{\frac{1}{2}(r-4)}},\quad \rho_2=\frac{v_2}{u_3^{\frac{1}{2}(r-4)}},\quad (l=4\ \ldots\ldots n,\ \ k=3\ \ldots\ldots n).$$

The remaining equations Wf in terms of σ, ρ are

$$\frac{\partial f}{\partial\rho_2}+\frac{\partial f}{\partial\rho_3}+\sum_4^n l(\sigma_l)^{-\alpha}\frac{\partial f}{\partial\rho_l}=0,\ [\alpha=\tfrac{1}{2}(r-4)],$$

$$W_t' f\equiv\frac{\partial f}{\partial\rho_3}+\sum_4^n l(\sigma_l)^{t-\alpha}\frac{\partial f}{\partial\rho_l}=0,\ (t=1,\ \ldots\ldots r-5).$$

Constructing a matrix,

$$\left\|\begin{array}{llllll}
\rho_2 & \rho_3 & \rho_4 & \ldots\ldots\ldots & \rho_{r-3+s}\ldots\ldots & \rho_n\\
1 & 1 & \sigma_4^{-\alpha} & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\\
0 & 1 & \sigma_4^{1-\alpha} & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\\
\ldots & \ldots & \ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\\
\ldots & \ldots & \ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\\
0 & 1 & \sigma_4^{r-5-\alpha} & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots
\end{array}\right\|$$

our invariants may be written

$$\sigma_l=\frac{x_2-x_l}{x_1-x_l}:\frac{x_2-x_3}{x_1-x_3},\ \text{and}\ \Delta_s,\ (l=4\ldots\ldots n,\ \ s=1\ldots\ldots n-r+3).$$

12. $$\boxed{\begin{array}{c} q,\quad xq,\quad x^2q,\ \ldots\ldots x^{r-5}q,\quad yq,\quad p,\quad xp,\quad x^2p+(r-5)xyq\\ r>5\end{array}}.$$

This group furnishes the system

$$W_t f\equiv\sum_1^n i\,x_i^t\frac{\partial f}{\partial y_i}=0,\ (t=0,\ 1\ldots\ldots r-5),\ \ Yf\equiv\sum_1^n i y_i\frac{\partial f}{\partial y_i}=0,$$

$$X_2 f \equiv \sum_1^n i \left(x_i^2 \frac{\partial f}{\partial x_i} + (r-5)\, x_i y_i \frac{\partial f}{\partial y_i}\right) = 0,$$

$$X_1 f \equiv \sum_1^n i x_i \frac{\partial f}{\partial x_i} = 0, \qquad X_0 f \equiv \sum_1^n i \frac{\partial f}{\partial x_i} = 0.$$

The first and last two equations show the final solutions to be functions of

$$u_k = \frac{x_1 - x_k}{x_1 - x_2}, \quad \psi_j = y_1 - y_j, \quad (k=3 \ldots\ldots n,\ j=2. \ \ldots . n).$$

$X_2 f$ now becomes

$$X_2' f \equiv \sum_3^n k u_k (u_k - 1) \frac{\partial f}{d u_k} + (r-5) \sum_3^n k u_k \frac{\partial f}{\partial \psi_k} + (r-5)\psi_2 \frac{\partial f}{\partial \psi_2} = 0,$$

with solutions

$$\sigma_l = \frac{u_3(u_l - 1)}{u_l(u_3 - 1)}, \quad \rho_k = \frac{\psi_k}{(u_k - 1)^{r-5}}, \quad \rho_2 = \psi_2 \left(\frac{u_3}{u_3 - 1}\right)^{r-5}, \ (l=4 \ \ldots\ldots n,\ k=3 \ \ldots\ldots n).$$

The remaining equations expressed in the new variables take the form

$$\left.\begin{aligned} &\frac{\partial f}{\partial \rho_2} + \frac{\partial f}{\partial \rho_3} + \sum_4^n l\, (\sigma_l)^{5-r} \frac{\partial f}{\partial \rho_l} = 0, \\ &W_t' f \equiv \frac{\partial f}{\partial \rho_3} + \sum_4^n l (\sigma_l)^{-t} \frac{\partial f}{\partial \rho_l} = 0, \quad (t=1, \ \ldots\ldots r-4). \end{aligned}\right\} (A).$$

$$Y' f \equiv \sum_2^n j\, \rho_j \frac{\partial f}{\partial \rho_j} = 0.$$

The independent determinants Δ_s, $(s=0, 1 \ldots . \ . n-r+3)$, of the matrix formed from equations (A) as in 11 will be solutions of (A). Δ_s will be linear in ρ, but $Y'f$ requires the ratios of ρ's to enter in the solutions. We may, therefore, write our invariants

$$\sigma_l = \frac{x_2 - x_l}{x_1 - x_l} \div \frac{x_2 - x_3}{x_1 - x_3},$$

the cross-ratios of the abscissas of four points, and

$$\theta_t = \frac{\Delta_t}{\Delta_0}, \quad (t=1 \ \ldots\ldots n - r + 3).$$

13. $\boxed{p, \ 2xp + yq, \ x^2 p + xyq}$

From this projective group we may obtain the complete system

$$\sum_1^n i\frac{\partial f}{\partial x_i}=\sum_1^n i\,(2x_i\frac{\partial f}{\partial x_i}+y_i\frac{\partial f}{\partial y_i})=\sum_1^n i\,(x_i^2\frac{\partial f}{\partial x_i}+x_iy_i\frac{\partial f}{\partial y_i})=0.$$

The solutions y_i and $\varphi_j=x_1-x_j$ of the first of these equations introduced in the last give

$$\sum_2^n j(\varphi_j^2\frac{\partial f}{\partial \varphi_j}+y_j\varphi_j\frac{\partial f}{\partial y_j})=0,$$

whose solutions are clearly

$$u_k=\frac{1}{\varphi_2}-\frac{1}{\varphi_k},\quad v_j=\frac{\varphi_j}{y_j},\quad y_1.$$

The second equation is now

$$y_1\frac{\partial f}{\partial y_1}-2\sum_3^n ku_k\frac{\partial f}{\partial u_k}+\sum_2^n jv_j\frac{\partial f}{\partial v_j}=0,$$

with solutions

$$\sigma_l=\frac{u_l}{u_3},\quad \rho_k=\frac{v_k}{v_2},\quad \rho_2=\frac{v_2}{y_1},\quad \rho_1=y_1u_3v_2,\quad (k=3\ldots\ldots n,\ l=4\ldots\ldots n).$$

Our invariants are then

$$\sigma_l=\frac{x_2-x_l}{x_2-x_3}:\frac{x_1-x_3}{x_1-x_l},\quad \rho_k=\frac{x_1-x_k}{x_1-x_2}:\frac{y_k}{y_2},\quad \rho_2=\frac{x_1-x_2}{y_1-y_2},\quad \rho_1=\frac{x_3-x_2}{x_3-x_1}:\frac{y_2}{y_1},$$

whose geometric significance is apparent.

14. $\boxed{yq,\quad p,\quad xp,\quad x^2p+xyq}$

This four-parameter projective group yields the complete system

$$\sum_1^n iy_i\frac{\partial f}{\partial y_i}=\sum_1^n i\frac{\partial f}{\partial x_i}=\sum_1^n ix_i\frac{\partial f}{\partial x_i}=\sum_1^n i(x_i^2\frac{\partial f}{\partial x_i}+x_iy_i\frac{\partial f}{\partial y_i})=0.$$

Here, we introduce the solutions,

$$\varphi_j=x_1-x_j,\quad \psi_j=\frac{y_j}{y_1},$$

of the first two equations in the last two as new variables, and have

$$\sum_1^n j\,\varphi_j\frac{\partial f}{\partial \varphi_j}=\sum_2^n j(\varphi_j^2\frac{\partial f}{\partial \varphi_j}+\varphi_j\psi_j\frac{\partial f}{\partial \psi_j})=0.$$

The last of these new equations is satisfied by

$$\sigma_k=\frac{1}{\varphi_1}-\frac{1}{\varphi_k},\quad \rho_j=\frac{\varphi_j}{\psi_j},$$

while the first becomes

$$\sum_3^n k\sigma_k\frac{\partial f}{\partial \sigma_k}-\sum_2^n j\rho_j\frac{\partial f}{\partial \rho_j}=0.$$

The invariants of this group are, therefore,

$$u_l=\frac{\sigma_l}{\sigma_3}=\frac{x_2-x_l}{x_2-x_3}:\frac{x_1-x_3}{x_1-x_l},\ (l=4\ldots\ldots n),$$

$$v_k=\frac{\rho_k}{\rho_2}=\frac{x_1-x_k}{x_1-x_2}:\frac{y_k}{y_2},\ (k=3\ldots\ldots n),$$

$$v_2=\sigma_3\,\rho_2=\frac{x_2-x_3}{x_1-x_3}:\frac{y_2}{y_1}.$$

§ 3.

Invariants of Such Imprimitive Groups as Leave Unchanged More Than One Family of ∞^1 Curves.

The groups of this section are classified according as they leave invariant:

(A) Two families of ∞^1 curves : x=constant, y=constant.

(B) A single infinity of families of ∞^1 curves : $ax+by$=constant.

(C) ∞^∞ of families of ∞^1 curves : $\varphi(x)+\psi(y)$=constant.

The calculations for the invariants of the remaining groups will be much abbreviated. The subscripts i, j, k, l, will be used to denote the series of numbers running from 1, 2, 3, 4, respectively, to n.

(A) *The families of curves x=constant, y=constant, remain invariant.*

15. $\boxed{q,\quad yq}$

The differential equations

$$\Sigma i\frac{\partial f}{\partial y_i}=\Sigma i\, y_i\frac{\partial f}{\partial y}=0,$$

belonging to this group, evidently leave invariant

$$X_i \text{ and } \psi_k=(y_1-y_k):(y_1-y_2).$$

16. $\boxed{q,\quad yq,\quad y^2q}$

This is the general projective group in one variable, and leaves invariant x_i, and the cross-ratios of any four ordinates. We may determine these cross-ratios by substituting ψ_k of 15 in

$$\Sigma i y_i^2\frac{\partial f}{\partial y_i}=0,$$

giving

$$\Sigma k\psi_k(\psi_k-1)\frac{\partial f}{\partial\psi_k}=0,$$

and integrating,

$$\rho_l=\frac{\psi_l-1}{\psi_l}:\frac{\psi_3-1}{\psi_3}=\frac{y_2-y_l}{y_2-y_3}:\frac{y_1-y_l}{y_1-y_3}.$$

17. $\boxed{q,\quad yq,\quad p}$

This group evidently leaves invariant

$$\psi_k=(y_1-y_k):(y_1-y_2), \text{ and } \varphi_j=x_1-x_j.$$

18. $\boxed{q, \quad yq, \quad y^2q, \quad p}$

The invariants for this group are clearly

$$\rho_l=\frac{y_2-y_l}{y_2-y_3}:\frac{y_1-y_l}{y_1-y_3}, \text{ as in 16 above, and}$$

$$\varphi_j=x_1-x_j, \text{ as in 17.}$$

19. $\boxed{q, \quad p, \quad xp+cyq}$

The solutions of the complete system corresponding to this group must be functions of $\psi_j=y_1-y_j$, $\varphi_j=x_1-x_j$, as shown by the first two members. These solutions, substituted in the last equation, give

$$\Sigma j\varphi_j\frac{\partial f}{\partial\varphi_j}+c\Sigma j\psi_j\frac{\partial f}{\partial\psi_j}=0,$$

with solutions

$$u_k=\frac{x_1-x_k}{x_1-x_2}, \quad v_k=\frac{y_1-y_k}{y_1-y_2}, \quad \sigma=\frac{(x_1-x_2)^c}{y_1-y_2}.$$

20. $\boxed{q, \quad yq, \quad p, \quad xp}$

The invariants here are

$$u_k=\frac{x_1-x_k}{x_1-x_2}, \quad v_k=\frac{y_1-y_k}{y_1-y_2}.$$

21. $\boxed{q, \quad yq, \quad y^2q, \quad p, \quad xp.}$

By this group

$$\rho_l=\frac{y_2-y_l}{y_2-y_3}:\frac{y_1-y_l}{y_1-y_3}, \quad u_k=\frac{x_1-x_k}{x_1-x_2}$$

remain invariant.

22. $\boxed{q, \quad yq, \quad y^2q, \quad p, \quad xp, \quad x^2p}$

This six-parameter group leaves invariant the cross-ratios of any four abscissas and ordinates :

$$\rho_l=\frac{y_2-y_l}{y_2-y_3}:\frac{y_1-y_l}{y_1-y_3}, \quad \sigma_l=\frac{x_2-x_l}{x_2-x_3}:\frac{x_1-x_l}{x_1-x_3}.$$

23. $\boxed{p+q, \quad xp+yq, \quad x^2p+y^2q}$

$$\Sigma\left(\frac{\partial f}{\partial x_i}+\frac{\partial f}{\partial y_i}\right)=\Sigma\left(x_i\frac{\partial f}{\partial x_i}+y_i\frac{\partial f}{\partial y_i}\right)=\Sigma\left(x_i^2\frac{\partial f}{\partial x_i}+y_i^2\frac{\partial f}{\partial y_i}\right)=0.$$

The solutions

$$\varphi_j = x_1 - x_j, \quad \psi_j = y_1 - y_j, \quad \sigma = x_1 - y_1$$

of the first equation, as new variables, give

$$\Sigma\left(\varphi_j \frac{\partial f}{\partial \varphi_j} + \psi_j \frac{\partial f}{\partial \psi_j}\right) + \sigma \frac{\partial f}{\partial \sigma} = \Sigma\left(\varphi_j^2 \frac{\partial f}{\partial \varphi_j} + \psi_j^2 \frac{\partial f}{\partial \psi_j}\right) + \sigma^2 \frac{\partial f}{\partial \sigma} = 0.$$

Selecting as solutions of the first of these new equations

$$u_k = \frac{\varphi_k}{\varphi_2}, \quad v_k = \frac{\psi_k}{\psi_2}, \quad \sigma_1 = \frac{\varphi_2}{\sigma}, \quad \sigma_2 = \frac{\psi_2}{\sigma},$$

we have

$$\Sigma\left(u_k(1-u_k)\frac{\partial f}{\partial u_k} + v_k(1-v_k)\frac{\partial f}{\partial v_k}\right) + \sigma_1(1-\sigma_1)\frac{\partial f}{\partial \sigma_1} + \sigma_2(1-\sigma_2)\frac{\partial f}{\partial \sigma_2} = 0.$$

Hence, our invariants may be selected as the cross-ratios

$$\sigma_l = \frac{u_l(1-u_3)}{u_3(1-u_l)} = \frac{x_2 - x_l}{x_2 - x_3} : \frac{x_1 - x_l}{x_1 - x_3}, \quad \rho_l = \frac{v_l(1-v_3)}{v_3(1-v_l)} = \frac{y_2 - y_l}{y_2 - y_3} : \frac{y_1 - y_l}{y_1 - y_3},$$

and the ratio

$$t = \frac{\sigma_1(1-\sigma_2)}{\sigma_2(1-\sigma_1)} = \frac{x_1 - x_2}{y_1 - y_2} : \frac{y_1 - x_2}{x_1 - y_2}.$$

(*B*) *All families of curves of the form* $ax + by =$ constant *remain invariant.*

24. $\boxed{q, \quad xp + yq}$

The first transformation of this group shows the differences $\psi_j = y_1 - y_j$ to enter in the final solutions. Hence the invariants may be written :

$$\rho_j = \frac{x_j}{x_1}, \quad u_k = \frac{y_1 - y_k}{y_1 - y_2}, \quad \sigma = \frac{y_1 - y_2}{x_1}.$$

25. $\boxed{p, \quad q}$

This group of translations leaves invariant

$$\varphi_j = x_1 - x_j, \quad \psi_j = y_1 - y_j.$$

26. $\boxed{p, \quad q, \quad xp + yq}$

By this group any homogeneous function of the differences $x_1 - x_j$, $y_1 - y_j$ will remain invariant.

$$u_k = \frac{x_1 - x_k}{x_1 - x_2}, \quad v_k = \frac{y_1 - y_k}{y_1 - y_2}, \quad \sigma = \frac{x_1 - x_2}{y_1 - y_2}.$$

(*C*) *The totality of curves* $\varphi(x) + \psi(y) =$ constant *remains invariant.*

27. $\boxed{q}$

This group is the only one of the class, and evidently leaves invariant the abscissas x_i, and the differences

$$\psi_j = y_1 - y_j.$$